Using the Roman numeral system

A simple way to let students think and present mathematics

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May 8, 2013

Introduction

We are taught early on how to add and subtract. Then a little while later we learn how to multiply and divide as well as work with fractions and exponentials. The arithmetic process is well-defined once the student learns their "tables." For addition, a student is expected to memorize — or more gently put, learn through repetition — all possible ways to add two single-digit numbers, and once that is known the addition process can be taught. Depending on the didactic methods, subtraction is taught similarly or is taught as "reverse addition." The same is done for multiplication and division.

Somewhere in a student's mathematical education, she is taught about the Roman numeral system, with their neat symbols — I, V, X, L, C, D and M. Then she is taught how to translate between the Hindu-Arabic numerals $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and the Roman numeral system. At this point, she is taught that certain numbers — 4, 9, 40, 90, 400, 900 — have a special representation. Respectively, that representation is IV, IX, XL, XC, CD, and CM. Finally, there are some exercises pertaining to this translation. Typically, that's where the lesson ends.

But what a moment we have here! Most students understand that + means "add" and it means to add in a very specific way. Abstractly, though, + is just an operator, whose operation is defined very specifically for the real numbers. Later students will learn that + takes on other meanings when dealing with different objects (fractions, complex numbers, vectors, matrices, etc.). If they take a programming course, they will learn about "operator overloading", wherein they will be able to define + for any object they create. But why wait all this time?

Why not have students define a rigorous process for $+, -, \times, \div$ using only Roman numerals? The only rule is that they cannot convert to another numeral system; they have to keep everything within Roman numerals.

Believe it or not, most students will struggle with this. Many will try in vain to come up with a nice mechanic for adding like we do with our numbering system. But those systems will fail when they are faced with something like XXIX + XIV. They'll convert it to 29 + 14 and resolve that to be 43 and then try to define a new logic so that the result will be XLIII. They'll try to introduce a way to "carry" or "borrow" (for subtraction) so that the operators work mechanically the same way they do for the Hindu-Arabic systems. But this isn't going to extend very nicely. It will, however, be a fun challenge.

Why would most students struggle with addition in Roman numerals even though they've been able to demonstrate that they can add? The will struggle because they never fully grasped addition. They've been able to follow the mechanic, but the underlying reasons have been lost. The same goes for subtraction, multiplication, and division.

So as a simple diversionary exercise, let's have our students try to come up with a rigorous arithmetic system for addition and subtraction using only Roman numerals. (The ambitious instructor can also have students come up with a system for multiplication and division. This is fairly straightforward to do and can make for an interesting larger project. This document, however, will keep focus on addition and subtraction.

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If there is reasonable demand for seeing how multiplication and division would work, feel free to contact me at help@mathmisery.com.) This will (hopefully) help do at least the following:

- Students will appreciate what we have today as opposed to what was done in antiquity (by one particular empire).
- This sets up further research into math history.
 - The instructor can explore other numbering systems (like the Babylonian system)
 - The instructor can ask students to wonder how Pythagoras would have represented his famous theorem relating the length of sides of a right triangle: $a^2 + b^2 = c^2$
- This will solidify for students what the basic arithmetic operations are.
- Students will get a first experience in having to explain their thoughts, processes, etc. Presentation skills are important!!
- As this is more open-ended than the usual "skills and drills", it will let students think openly and creatively since the answer is a *process* and not a number or formula.

No instructor should view this (or any) document as a canned lesson plan. Every instructor should tailor their instruction to best meet the needs of their students.

The instructor should be open to the many ideas that students may have. Not all of them will be correct, in fact, many will be well off-the-mark. The further off-the-mark, the more teachable moments there are, so welcome the zaniest ideas and discuss them! Also, there may be more than one way to solve things and this document is not going to try to explore all possibilities. So the instructor should not hold rigidly that "this is how it's done." We are good instructors and we let students think for themselves.

So, let's ask our students, "How would the Romans have done it?"

How would the Romans have done it?

When doing this exercise with students, the instructor should give plenty of examples and motivation. The worst thing that an instructor can do is make this into yet another painful set of drills. That is not the point of this!

Some things to note are as follows:

- Make sure that students are comfortable with the "special" representations for example, 9 = IX.
- Make sure that students can readily read LXIX as 49.
- Keep the examples and sample problems tame but instructive. For example, don't overwhelm students by having them subtract 3473 by 82 (MMMCDLXXIII LXXXII). Instead do something like 64 19 (LXIV XIX) or 43 + 12 (XLIII + XII). The first is probably a bit too cumbersome, while the last two are ones that students can work out with ease on their own using the Hindu-Arabic numeral system to verify their answer in the Roman numeral system.

Addition

When introducing addition in the Roman numeral system, ask students to resolve some of the elementary cases first. How would the Romans have added

- III + I?
- III + II?
- V + II?

- X + V?
- X + II?
- IV + I?
- IX + II?

Some students will undoubtedly remark that they have a method for addition. They may argue as follows:

Since III + I is just 3 + 1, we know that should be 4, but we can't write that as IIII. Instead, we would write IV. The same thing would go for III + II which would be V because we can't write IIIII. And you see this also works for V + II because we just combine the numerals and get VII which is the answer. The same thing goes for X + V and X + II.

So far so good ...

But, when we want to add IV + I we just get rid of the I from each number and get V. And you see the same thing happens when we do it with IX + II. If we get rid of one I from each of the numbers it's the same thing as X + I which is XI and that's what 9 + 2 is.

And this is exactly a result of a mechanical understanding of addition in the Hindu-Arabic numeral system. The student is trying to mimic some form of "carry". To help make this point, a next question to ask the student is

How would that methodology add IX + IV? If we removed the I from each number we would get X + V = XV. Is that correct?

Clearly, $9 + 4 = 13 = XIII \neq XV$.

At this point, the instructor should allow the students to make the connection that IV or IX is just symbolism for IIII or VIIII. Thus, IV + I should first be written as IIII + I. This gives IIIII and that, by definition is V.

Similarly,

$$\begin{split} \mathrm{IX} + \mathrm{IV} &= \mathrm{VIIII} + \mathrm{IIII} \\ &= \mathrm{V} + \mathrm{IIIIIIII} \\ &= \mathrm{V} + \mathrm{IIIII} + \mathrm{III} \\ &= \mathrm{V} + \mathrm{V} + \mathrm{III} \\ &= \mathrm{VV} + \mathrm{III} \\ &= \mathrm{X} + \mathrm{III} \\ &= \mathrm{XIII} \end{split}$$

It is vitally important that the student understand this. It's all about grouping. Putting things of the "same kind" together and then using the appropriate symbolism to resolve things like VV.

For the instructor, the point of this is not to teach them a new mechanic but to get them to understand what exactly was wrong with the logic of removing I from each number, for example. If the student truly understands this, then the instructor should be able to change the theme and the student should be able to correctly create a generally rigorous process for addition that is not filled with exceptions.

Try giving students these to resolve:

- XIX + XXIV = ?
- XLIX + L = ?
- XXXIV + XXIV =?

Subtraction

If the students have truly grasped how addition works, subtraction should be no problem. Though students will have to take some care in keeping their books in order. Let's start simple again. I will also keep it that we don't have to run into negative numbers. Though, that shouldn't deter the instructor from going the extra step. These should be no sweat.

- III I =?
- VIII II =?
- XVI V = ?

Now let's turn the dial up one setting.

- VI II = ?
- XXII III = ?
- XIX IV =?

For these sets of problems, the student should have recognized from the addition section to make "like terms". Here is a step-by-step process. Remember, it's important that students be able to explain each step, not just plug away like mindless automatons.

XLIV - XIX = XXXXIIII - XVIIII first we get rid of all "special" forms XXXXIIII - XVIIII = XXXXIIII - XVIIII next take away all things common from both numbers = XXX - V rewrite a bit more cleanly = XXVV - V break up one of the Xs into two Vs = XXVV - V take away once again = XXV and we're done!

So, what should students be able to do here? Unlike our system of "borrowing", subtraction in Roman numerals would be an iterative process until one of the groups of numerals "goes away". If the first group goes away before the second, then the quantity is negative. If the second group goes away before the first, the quantity is positive. If both groups go away simultaneously, then we have zero (but there's no way to write zero in the classical Roman numeral system! (Hint, hint: a math history teaching moment)).

Have students compare how they would subtract 19 from 44 versus how it would be done if they had to use Roman numerals. Which one do they find more intuitive? Which process do they find more efficient? Personally, I find subtraction in Roman numerals more intuitive, though clearly the process is much more tedious.

Summary

Hopefully, you and your students will enjoy this little excursion.

I have found this to be a fun exercise, especially, when students can work in groups and present their results to the class. The emphasis isn't really on what's right or wrong, but on the thought process they used to arrive at their method. As instructors we can drive home the point that learning is an iterative process and that when we have to tackle an open problem we have to be willing to try all sorts of things. Simultaneously though, we have to try in an organized way. Shooting in the dark, may hit the target, but we're going to waste a lot of bullets. So, similarly, with this exercise (and maybe any exercise), the first thing students should do is to take some time to put their feet up, stare at the ceiling, and think about the concepts before they work out the mechanics.

If we can do this, we've opened up our students to learning.